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Connections between Kelvin functions and zeta functions with applications

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Abstract. A very simple method for resumming infinite series of the type

$$\sum_{l=1}^{\infty} \frac{(\pm 1)^{l+1}}{l^{(n+1)/2}} \cosh(la) K_{(n+1)/2}(lb)$$

is presented. As a result a series expansion in powers of a and b is obtained, relevant to many physical applications.

1. Introduction

Using the zeta-function definition of functional determinants given first by Dowker and Critchley [1] and Hawking [2], the calculation of effective actions in different contexts [3–12] and the consideration of the Casimir energy for several configurations [13–19] leads to series of Kelvin functions of the type

$$\sum_{l=1}^{\infty} \frac{c^{l+1}}{l^{(n+1)/2}} \cosh(la) K_{(n+1)/2}(lb) \tag{1.1}$$

or multi-dimensional generalizations of this, where $c = \pm 1$, $n \in \mathbb{N}$, $a, b \in \mathbb{R}$, and we assume $b > |a|$ for absolute convergence.

For example in five dimensional Kaluza–Klein theories [4–8] ($c = 1$, $a = 0$), the parameter b is connected with the circumference of the compact fifth dimension. In the calculation of the Casimir energy [18] ($c = 1$, $a = 0$), b is proportional to the product of a mass and a compactification length or plate separation. Finally, in the context of finite temperature quantum field theory of a free charged massive relativistic bose gas ($c = 1$) or spin-$\frac{1}{2}$ gas ($c = -1$) [9], b (respectively a) is proportional to the product of the inverse temperature with the mass of the field (respectively the chemical potential of a conserved charge).

The most important property of the series (1.1) is that it is very useful in the range $b \gg a$, because the Kelvin functions are exponentially damped for $b \to \infty$ and so (1.1) converges very quickly. Often one would like to represent equation (1.1)

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in form of a power series in a and b, valid in the range $b \ll 1$, where (1.1) is not suitable.

For example in finite temperature quantum field theory the range $b \ll 1$ means the high-temperature limit of the theory, which has been considered intensively in the context of Bose–Einstein condensation of relativistic gases [20–22]. In order to find the high-temperature expansion of the free-energy or the grand thermodynamic potential of the spin-0 or spin-$\frac{1}{2}$ gas many different methods and techniques have been employed [9, 23–28]. One possibility is to use the series representation of the Kelvin functions and the hyperbolic functions and to resum the resulting expressions [9]. The relevant resummation theorems have been derived by Weldon [29] and corrections have been given by Elizalde and Romeo [30], where additional contributions of contour integrals have been found. These resummation theorems have been used furthermore to calculate the Casimir energy of a scalar field with mass M in the space $S^1 \times \mathbb{R}^{n-1}$, compactification length L, in the range $ML \ll 1$ [18]. The question at which value of $ML \ll 1$ contour integral contributions become significant is formulated.

The aim of this paper is to obtain a power series expansion of

$$\sum_{l=1}^{\infty} \frac{c^{l+1}}{l^{(1/2)-s}} \cosh(la) K_{(1/2)-s}(lb) \tag{1.2}$$

in powers of a and b for arbitrary values of $s \in \mathbb{C}$ using a very simple derivation where no reference to the mentioned resummation theorems is made. The simplicity of this derivation and the generality of the result is the main motivation for presenting these considerations.

Starting from the generalized inhomogeneous Epstein zeta function

$$Z_1^{m^2}(s; d; z) := \sum_{n=-\infty}^{\infty} \frac{1}{[d(n+z)^2 + m^2]^s} \tag{1.3}$$

$m \in \mathbb{R}$, $d \in \mathbb{R}$, $d > 0$, $z \in \mathbb{C}$, $m^2 + d\,\mathrm{Re}(z^2) > 0$, valid for $\mathrm{Re}(s) > 1/2$, it is shown in section 2 that using a Mellin transformation the analytic continuation of $Z_1^{m^2}(s; 1; z)$ to $\mathrm{Re}(s) < 1/2$ may be given in terms of equation (1.2) (for similar considerations see [31]). On the other side it is possible to analytically continue equation (1.3) by expanding $Z_1^{m^2}(s; 1; z)$ in powers of m^2 [19, 23]. Of course the resulting expressions obtained by two different analytic continuation methods are equal by construction and with the replacement $z = \frac{ia}{2\pi}$, $m = \frac{b}{2\pi}$, comparison yields the expansion of equation (1.2) in powers of a and b we are looking for. The series is convergent and the radius of convergence is determined; in the context of the Casimir energy this yields the range of values where the contour integral does not contribute.

In section 3 physical applications of the result derived in section 2 are given. In particular, the calculation of the Casimir energy for a massive non-interacting bosonic field in $S^1 \times \mathbb{R}^{n-1}$ [18] and the derivation of the high-temperature expansion of the grand thermodynamic potential of a free massive spin-0 and spin-$\frac{1}{2}$ gas is given in a very compact form. In the context of the grand thermodynamic potential the given considerations provide the connection between the different methods presented for example in [9] and [23].

2. Inhomogeneous Epstein-zeta functions

The aim of this section is to derive a power series expansion of (1.2) in powers of a and b for any values of $s \in \mathbb{C}$.

Consider the generalized Epstein-zeta-function

$$Z_1^{m^2}(s;1;z) := \sum_{n=-\infty}^{\infty} \left[(n+z)^2+m^2\right]^{-s} \tag{2.1}$$

$m \in \mathbb{R}$, $m > 0$, where the given definition is valid for $\mathrm{Re}(s) > \frac{1}{2}$ and $m^2+\mathrm{Re}(z^2) > 0$. Using a Mellin-transformation and employing for $t \in \mathbb{R}_+$, $\nu \in \mathbb{C}$

$$\sum_{n=-\infty}^{\infty} \exp\left\{-tn^2+2\pi i n\nu\right\} = \left(\frac{\pi}{t}\right)^{1/2} \sum_{n=-\infty}^{\infty} \exp\left\{-\frac{\pi^2}{t}(n-\nu)^2\right\} \tag{2.2}$$

which is due to Jacobi's relation between theta functions [32], an analytical continuation of $Z_1^{m^2}(s;1;z)$ with respect to s in terms of Kelvin-functions may be found with the result

$$\begin{aligned} Z_1^{m^2}(s;1;z) &= \frac{\sqrt{\pi}}{\Gamma(s)}\frac{1}{m^{2s-1}} \\ &\quad \times \left\{\Gamma\left(s-\frac{1}{2}\right) + 4\sum_{l=1}^{\infty} \frac{\cos(2\pi l z)}{(\pi l m)^{(1/2)-s}} K_{(1/2)-s}(2\pi l m)\right\}. \end{aligned} \tag{2.3}$$

So $Z_1^{m^2}(s;1;z)$ is seen to be a meromorphic function with respect to s with poles of order 1 at $s = \frac{1}{2} - n$, $n \in \mathbb{N}_0$.

On the other side it is possible to expand (2.1) in powers of m^2 [19, 23], which yields

$$Z_1^{m^2}(s;1;z) = \left(\frac{1}{m^2+z^2}\right)^s + \sum_{n=-\infty}^{\infty}{}' \sum_{l=0}^{\infty} (-1)^l \frac{\Gamma(s+l)}{l!\,\Gamma(s)} m^{2l}(n+z)^{-2s-2l} \tag{2.4}$$

where the prime means omission of the summation index $n = 0$. For $m^2 < 1$ the series is absolutely convergent and interchanging the n and l summation one finds the result

$$\begin{aligned} Z_1^{m^2}(s;1;z) &= \left(\frac{1}{m^2+z^2}\right)^s + \sum_{l=0}^{\infty}(-1)^l \frac{\Gamma(s+l)}{l!\,\Gamma(s)} m^{2l} \\ &\quad \times \left\{\zeta_H(2s+2l;1+z) + \zeta_H(2s+2l;1-z)\right\} \end{aligned} \tag{2.5}$$

with the Hurwitz-zeta function

$$\zeta_H(\nu;v) := \sum_{n=0}^{\infty} \frac{1}{(n+v)^s}.$$

Just by comparison of (2.3) with (2.5) the very interesting equation

$$\sum_{l=1}^{\infty} \frac{\cos(2\pi l z)}{(\pi l m)^{(1/2)-s}} K_{(1/2)-s}(2\pi l m)$$
$$= \frac{m^{2s-1}}{4\sqrt{\pi}} \sum_{l=0}^{\infty} (-1)^l \frac{\Gamma(s+l)}{l!} m^{2l}$$
$$\times \{\zeta_H(2s+2l;1+z) + \zeta_H(2s+2l;1-z)\}$$
$$+ \frac{m^{2s-1}\Gamma(s)}{4\sqrt{\pi}(z^2+m^2)^s} - \frac{\Gamma(s-1/2)}{4} \tag{2.6}$$

valid for $0 < m < 1$, $m^2 + \mathrm{Re}(z^2) > 0$, is derived. Obviously, the left hand side is an analytical function with repect to s. For $s = \frac{1}{2} - N$ and $s = -N$, $N \in \mathbb{N}_0$, individual terms on the right-hand side have poles of order one which cancel by construction. So for these values of s the right-hand side is to be understood as the analytical continuation to $s = \frac{1}{2} - N$, respectively $s = -N$, which always yields finite results.

Using the Taylor series representation of $\zeta_H(\nu; 1+c)$ [33]

$$\zeta_H(\nu;1+c) = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\nu+k)}{k!\Gamma(\nu)} \zeta_R(\nu+k) c^k$$

valid for $|c| < 1$, the expansion we are looking for is derived

$$\sum_{l=1}^{\infty} \frac{\cos(2\pi l z)}{(\pi l m)^{(1/2)-s}} K_{(1/2)-s}(2\pi l m)$$
$$= \frac{m^{2s-1}}{2\sqrt{\pi}} \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} (-1)^l \frac{\Gamma(s+l)\Gamma(2s+2l+2k)}{l!(2k)!\Gamma(2s+2l)} \zeta_R(2s+2l+2k) z^{2k} m^{2l}$$
$$+ \frac{m^{2s-1}\Gamma(s)}{4\sqrt{\pi}(z^2+m^2)^s} - \frac{\Gamma\left(s-\frac{1}{2}\right)}{4}. \tag{2.7}$$

The second term on the right-hand side shows the necessity of the condition $m^2 + \mathrm{Re}(z^2) > 0$.

The analogous representation for the series with alternating sign is easily found by noting that

$$\sum_{l=1}^{\infty} (-1)^{l+1} \frac{\cos(2\pi l z)}{(\pi l m)^{(1/2)-s}} K_{(1/2)-s}(2\pi l m)$$
$$= \sum_{l=1}^{\infty} \frac{\cos(2\pi l z)}{(\pi l m)^{(1/2)-s}} K_{(1/2)-s}(2\pi l m)$$
$$- 2\sum_{l=1}^{\infty} \frac{\cos(4\pi l z)}{(2\pi l m)^{(1/2)-s}} K_{(1/2)-s}(4\pi l m). \tag{2.8}$$

The Taylor series expansion in powers of m and z for the series with alternating sign then reads

$$\sum_{l=1}^{\infty}(-1)^{l+1}\frac{\cos(2\pi lz)}{(\pi lm)^{(1/2)-s}}K_{(1/2)-s}(2\pi lm)$$
$$= \frac{m^{2s-1}}{2\sqrt{\pi}}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}(-1)^l\frac{\Gamma(s+l)\Gamma(2s+2l+2k)}{l!(2k)!\Gamma(2s+2l)}\zeta_R(2s+2l+2k)$$
$$\times\left(1-2^{2l+2s+2k}\right)z^{2k}m^{2l}+\frac{1}{4}\Gamma\left(s-\frac{1}{2}\right) \tag{2.9}$$

exact for $0 < m < \frac{1}{2}$ and $|z| < \frac{1}{2}$.

Let us again stress, that (2.7) and (2.9) are valid for arbitrary $s \in \mathbb{C}$, where for $s = \frac{1}{2} - N$ and $s = -N$, $N \in \mathbb{N}_0$, the right-hand-side is to be understood as the analytical continuation to these values.

In the context briefly described in the Introduction, the values $s = -N$ and $s = \frac{1}{2} - N$ are of particular interest. The corresponding results are easily found by considering (2.7) and (2.9) at $s = -N + \epsilon$, respectively $s = -N + \frac{1}{2} + \epsilon$, and to use Taylor or Laurent series expansions around $\epsilon = 0$ of the involved functions [34]. In the limit $\epsilon \to 0$ the following special cases of (2.7) are found:

(i) $s = \frac{1}{2} - N$

$$\sum_{l=1}^{\infty}\frac{\cos(2\pi lz)}{(\pi lm)^N}K_N(2\pi lm)$$
$$= \frac{1}{2(\pi m)^{2N}}\sum_{l=0}^{N-1}\sum_{k=0}^{N-l}(-1)^{k+l}$$
$$\times\frac{(N-l-1)!}{l!(2k)!}(\pi m)^{2l}(2\pi z)^{2k}\zeta_R(2N-2l-2k)$$
$$+\frac{(-1)^N}{2}\sum_{l=1}^{\infty}\sum_{k=0}^{\infty}(-1)^l\frac{(2l+2k)!}{l!(l+N)!(2k)!}\left(\frac{m}{2}\right)^{2l}z^{2k}\zeta_R(1+2l+2k)$$
$$+\frac{\Gamma(-N+1/2)}{4\sqrt{\pi}m^{2N}}\left(m^2+z^2\right)^{N-1/2}$$
$$+\frac{(-1)^N}{4N!}\left\{\psi\left(\frac{1}{2}\right)-\psi(N+1)-\psi(z)-\psi(-z)+2\ln m\right\} \tag{2.10}$$

(ii) $s = -N$

$$\sum_{l=1}^{\infty}\frac{\cos(2\pi lz)}{(\pi lm)^{1/2+N}}K_{1/2+N}(2\pi lm)$$
$$= \frac{(-1)^N}{2\sqrt{\pi}m}\sum_{l=1}^{\infty}\sum_{k=0}^{\infty}(-1)^l\frac{(l-1)!(2l+2k-1)!}{(l+N)!(2k)!(2l-1)!}\zeta_R(2l+2k)m^{2l}z^{2k}$$
$$+\frac{(-1)^N}{2\sqrt{\pi}m}\sum_{l=0}^{N}\sum_{\substack{k=0\\k\neq l}}^{\infty}(-1)^{l+k}$$
$$\times\frac{(2l)!}{(N-l)!l!(2k)!}(2\pi m)^{-2l}(2\pi z)^{2k}\zeta_R(1+2l-2k)+$$

$$
\begin{aligned}
&+\frac{(-1)^{N+1}}{4\sqrt{\pi}m}\sum_{l=0}^{N}\frac{m^{-2l}z^{2l}}{(N-l)!l!}\left[\psi(l+1)-2\psi(2l+1)\right]\\
&+\frac{(-1)^N}{N!}\frac{1}{4\sqrt{\pi}m^{2N+1}}\left(z^2+m^2\right)^N\\
&\times\left[\psi(N+1)+2\gamma-\ln\left\{4\pi^2\left(z^2+m^2\right)\right\}\right]\\
&-\frac{1}{4}\Gamma\left(-N-\frac{1}{2}\right)
\end{aligned}
\tag{2.11}
$$

and for (2.9) they read:

(iii) $s=\frac{1}{2}-N$

$$
\begin{aligned}
\sum_{l=1}^{\infty}(-1)^{l+1}\frac{\cos(2\pi lz)}{(\pi lm)^N}K_N(2\pi lm)
&=\frac{1}{2(\pi m)^{2N}}\sum_{l=0}^{N-1}\sum_{k=0}^{N-l}(-1)^{k+l}\frac{(N-l-1)!}{l!(2k)!}\\
&\times(\pi m)^{2l}(2\pi z)^{2k}\eta(2N-2l-2k)\\
&+\frac{(-1)^N}{2}\sum_{l=1}^{\infty}\sum_{k=0}^{\infty}(-1)^l\frac{(2l+2k)!}{l!(l+N)!(2k)!}\\
&\times\left(\frac{m}{2}\right)^{2l}z^{2k}\left[1-2^{1+2l+2k}\right]\zeta_R(1+2l+2k)\\
&+\frac{(-1)^N}{4N!}\times\left\{\psi(N+1)+\gamma-2\ln\left(\frac{m}{2}\right)+\psi\left(\frac{1}{2}+z\right)+\psi\left(\frac{1}{2}-z\right)\right\}
\end{aligned}
\tag{2.12}
$$

(iv) $s=-N$

$$
\begin{aligned}
\sum_{l=1}^{\infty}(-1)^{l+1}\frac{\cos(2\pi lz)}{(\pi lm)^{(1/2)+N}}K_{(1/2)+N}(2\pi lm)
&=\frac{(-1)^N}{2\sqrt{\pi}m}\sum_{l=1}^{\infty}\sum_{k=0}^{\infty}(-1)^l\frac{(l-1)!(2l+2k-1)!}{(l+N)!(2k)!(2l-1)!}\\
&\times\zeta_R(2l+2k)m^{2l}z^{2k}\left[1-2^{2l+2k}\right]\\
&+\frac{(-1)^N}{2\sqrt{\pi}m}\sum_{l=0}^{N}\sum_{\substack{k=0\\k\neq l}}^{\infty}(-1)^{l+k}\frac{(2l)!}{(N-l)!l!(2k)!}\\
&\times(2\pi m)^{-2l}(2\pi z)^{2k}\eta(1+2l-2k)\\
&+\frac{(-1)^N\ln 2}{2\sqrt{\pi}m^{2N+1}N!}\left(m^2+z^2\right)^N+\frac{1}{4}\Gamma\left(-N-\frac{1}{2}\right)
\end{aligned}
\tag{2.13}
$$

where $\psi(z)=\Gamma'(z)/\Gamma(z)$ and $\eta(s)=\left(1-2^{1-s}\right)\zeta_R(s)$. These results are, of course, in agreement with the results derived for example in [18, 11] using different methods. In my opinion it is satisfying to see, that equations (2.10)–(2.13) may be summarized in (2.7) and (2.9) depending on a complex parameter s, which in the physical examples interpolates the dimension of spacetime.

3. Physical applications

In this section we want to give some physical applications of the results derived in section 2. The examples considered have already been discussed in the literature using resummation techniques [9, 11, 18] so our description will be very brief.

Consider first the evaluation of the Casimir energy for a massive non-interacting bosonic field being defined in the space $S^1 \times \mathbb{R}^{n-1}$. Without repeating the analysis given in [18] (see equation (2.9) of that reference), the Casimir energy density $\epsilon(n, L, M)$, depending on the mass M of the field, the compactification length L and the dimension n, may be given in the form

$$\epsilon(n, L, M) = -\frac{2}{\pi^{(n+1)/2} L^{n+1}} S(n, L, M) \tag{3.1}$$

where

$$S(n, L, M) = \sum_{l=1}^{\infty} \frac{1}{l^{n+1}} \left(\frac{LMl}{2}\right)^{(n+1)/2} K_{(n+1)/2}(LMl). \tag{3.2}$$

In [18], a series expansion in powers of LM has been obtained by making use of the resummation theorems derived in [29, 30]. In that approach a discussion of contributions of contour integrals is necessary in order to determine for which values of LM the series is exact (see appendix B of [18]). Another method is provided in section 2 and inserting (2.10) (respectively (2.11))) with $c = 0$, $m = (\frac{LM}{2\pi})$ and $n = 2N - 1$ (respectively $n = 2N$) in (3.2), the series expansion for the energy $\epsilon(n, L, M)$ is found (see equations (3.12) and (3.19) of [18]). The advantage of the described method is that no contributions of contour integrals have to be discussed. As argued in section 2 the results are exact for $0 \leqslant \frac{LM}{2\pi} < 1$. In the context of [18] this means that the corrections due to contour integrals will vanish in this range.

Let us now briefly mention the grand thermodynamic potential Ω_B, Ω_F, of a massive charged non-interacting spin-0 or spin-$\frac{1}{2}$ gas. The starting point for these considerations are the integrals [9, 20]

$$\Omega_B(\beta, M, \mu) = \frac{TV}{(2\pi)^N} \int \mathrm{d}^N k \left[\ln\left[1 - \exp\left\{-\beta\left(\sqrt{k^2 + M^2} + \mu\right)\right\}\right] + \mu \to -\mu\right] \tag{3.3}$$

for the spin-0 gas and

$$\Omega_F(\beta, M, \mu) = -\frac{d}{2}\frac{TV}{(2\pi)^N} \times \int \mathrm{d}^N k \left[\ln\left[1 + \exp\left\{-\beta\left(\sqrt{k^2 + M^2} + \mu\right)\right\}\right] + \mu \to -\mu\right] \tag{3.4}$$

for the spin-$\frac{1}{2}$ gas, where V is the spatial volume, T is the temperature, $\beta = \frac{1}{T}$, d is the dimension of the Dirac representation of the γ-matrices, μ is a chemical potential associated with a conserved charge and $N+1$ is the dimension of spacetime. It has been shown in [9], that equations (3.3) and (3.4) may be written as

$$\Omega_B(\beta, M, \mu) = -4VT^{N+1}\sum_{l=1}^{\infty}\frac{1}{l^{N+1}}\left(\frac{l\beta M}{2\pi}\right)^{(N+1)/2} K_{(N+1)/2}(l\beta M)\cosh(l\beta\mu)$$

$$\Omega_F(\beta, M, \mu) = -2dVT^{N+1}\sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l^{N+1}}\left(\frac{l\beta M}{2\pi}\right)^{(N+1)/2} \times K_{(N+1)/2}(l\beta M)\cosh(l\beta\mu).$$

With the replacements $m = \frac{\beta M}{2\pi}$ and $z = \mathrm{i}\frac{\beta\mu}{2\pi}$, (2.10)–(2.13) provide the high-temperature expansion of Ω_B and Ω_F. The results are well known and need not to be repeated here (see [9, 23]). As we have shown the results for $\Omega_B(\beta, M, \mu)$ are exact for $\mu^2 < M^2$ (which is the standard restriction in finite temperature relativistic field theory) and $\left(\frac{\beta M}{2\pi}\right)^2 < 1$, whereas the results for $\Omega_F(\beta, M, \mu)$ are exact for $\left(\frac{\beta\mu}{\pi}\right)^2 < 1$ and $\left(\frac{\beta M}{\pi}\right)^2 < 1$.

4. Conclusions

The resummation theorems proven in [29, 30] have a wide range of applicability (see for example in [9, 11, 16–18, 29, 30, 35, 36]). One main motivation to look for them was to find the high-temperature expansion of the grand thermodynamic potential of a non-interacting massive bosonic or fermionic field.

However, as we have seen the resummation of series over Kelvin functions may also be obtained by a different and very simple method of analytical continuation of generalized Epstein-zeta functions. Equations (2.7) and (2.9) are the main results of this paper, important for example in the context of the physical applications which we have briefly described.

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